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The Effect of $w - term$ on Visibility Correlation and Power Spectrum Estimation

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ABSTRACT

Visibility-visibility correlation has been proposed as a technique for the estimation of power spectrum, and used extensively for small field of view observations, where the effect of $w - term$ is usually ignored. We consider power spectrum estimation from the large field of view observations, where the $w - term$ can have a significant effect. Our investigation shows that a nonzero w manifests itself as a modification of the primary aperture function of the instrument. Using a gaussian primary beam, we show that the modified aperture is an oscillating function with a gaussian envelope. We show that the two visibility correlation reproduces the power spectrum beyond a certain baseline given by the width, U_w of the modified aperture. Further, for a given interferometer, the maximum U_w remains independent of the frequencies of observation. This suggests that, the incorporation of large field of view in radio interferometric observation has a greater effect for larger observing wavelengths.

Key words: cosmology:observations-method:observational-technique:interferometric

1 INTRODUCTION

The directly observed quantity in radio interferometry is the complex visibility, measured as a function of baseline and frequency. Apart from being the building blocks for radio imaging (Perley et al., 1989), the observed visibility data can be used for the estimation of various statistical properties of the radio signal. Visibility based estimators have been widely used to estimate the

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power spectrum of intensity fluctuations of the observed radio signal (Bharadwaj & Sethi, 2001; Bharadwaj & Ali, 2005; Dutta et al., 2009a) and have also been proposed to be a viable probe of Bispectrum (Saiyad Ali et al., 2006; Dutta et al., in preperation). For the Cosmic Microwave Background Radiation (CMBR) signal, the formalism Hobson & Maisinger (2002) shows the effectiveness of using Maximum Likelihood Estimators (MLE) in the Visibility space for direct estimation of the angular power spectrum. A large number of radio telescopes, like the presently functioning GMRT [1], upcoming MWA [2] and SKA [3] aim to probe the universe through the redshifted 21 cm observations. The HI power spectrum is an unique observational probe of the dark ages (Bharadwaj & Sethi, 2001), the epoch of reionization (Datta et al., 2007) and the era, post-reionization (Bharadwaj et al., 2009; Saiyad Ali et al., 2006). The visibility based estimator has advantages over the image based methods for the estimation of power spectrum. Apart from the fact that it uses the data in its raw form, visibility based method is naturally more useful in cases of incomplete/sparse u-v coverage. The method has also been used on smaller galactic scales for estimation of HI power spectrum of a turbulent ISM (Dutta et al., 2008, 2009a,b) and continuum power spectrum of Supernova remnant (Roy et al., 2008).

The definition of visibility $\mathcal{V}$ is often approximated as a 2D Fourier transform of the sky intensity distribution i.e, $\mathcal{V}(u,v) \overset{FT}{\rightleftharpoons} I(l,m)$, where (l,m) denote the two angular coordinates in the sky and (u,v), the corresponding Fourier conjugate variables. While, this is true for small field of view, and co-planar array distribution, the actual 3D visibility as is observed in radio interferometers, do not satisfy this simple relationship. For most practical purposes, at high observing frequencies, this approximation works reasonably well for the existing radio interferometers like the GMRT , VLA etc. However, upcoming radio telescopes aiming at wide field imaging will require inclusion of the full 3D effect. The same problem is also faced in imaging for large field of view observations or observations with non-coplanar baselines and is tackled by techniques of mosaicing (Perley et al., 1989; Frater & Docherty, 1980; BCornwell & Perley, 1992; Cornwell et al., 2008).

However the problem of estimating power spectra using the measured three dimensional visibility $\mathcal{V}(u,v,w)$, has not been addressed.

In this letter we investigate the effect of using the three dimensional visibility in the estimation of power spectrum and hence find a possible justification of the simpler two dimensional approach used extensively earlier.

[1] http://www.gmrt.ncra.tifr.res.in/

[2] http://www.mwatelescope.org/

[3] http://www.skatelescope.org/

2 EFFECT OF $w - term$ ON THE APERTURE FUNCTION

The direct observable in the radio interferometric observations is the complex visibility $\mathcal{V}^{3D}(\vec{U}, \nu)$. For a pair of antennae separated by $\vec{d}$, with each antenna pointing along the direction of the unit vector $\hat{k}$ (referred to as the phase center) we have

$$\mathcal{V}^{3D}(\vec{U}, \nu) = \int d\Omega_{\hat{n}} e^{2\pi \vec{U}\cdot(\hat{n}-\hat{k})} A(\hat{n} - \hat{k}, \nu) I(\hat{n} - \hat{k}, \nu) \tag{1}$$

where $\hat{n}$ denotes the unit vector to different directions of the sky, baseline $\vec{U} = \vec{d}/\lambda$, $A(\hat{n} - \hat{k}, \nu)$ denotes the primary beam and $I(\hat{n} - \hat{k}, \nu)$ is the specific intensity. Writing $\vec{U} = \vec{U}_\perp + w\hat{k}$, where $\vec{U}_\perp$ is a 2D vector, and defining $\hat{n} - \hat{k} = \vec{\theta}$, we have, for $|\vec{\theta}| \ll 1,\ \vec{\theta} \cdot \hat{k} = 0$, implying that $\vec{\theta}$ is a $2D$ vector. In this limit $\vec{\theta}$ gives the position of any point on the sky with respect to the phase centre in a 2D tangent plane. This is known as the flat-sky approximation. The term $w\hat{k}$ quantifies deviation from this.

In the $2D$ approximation we have

$$\mathcal{V}^{2D}(\vec{U}_\perp, \nu) = \int d^2\vec{\theta}\, e^{2\pi i \vec{U}_\perp \cdot \vec{\theta}}\, A(\vec{\theta}, \nu)\, I(\vec{\theta}, \nu) \tag{2}$$

Writing the specific intensity $I(\vec{\theta}, \nu)$ as $I(\vec{\theta}, \nu) = \bar{I}_\nu + \delta I(\vec{\theta}, \nu)$, where the first term is a constant background and the second term is a fluctuation, we have

$$\mathcal{V}^{2D}(\vec{U}_\perp, \nu) = \bar{I}_\nu \tilde{A}(\vec{U}_\perp, \nu) + \tilde{A}(\vec{U}_\perp, \nu) \otimes \tilde{\delta I}(\vec{U}_\perp, \nu) \tag{3}$$

where tilde represents a Fourier transform and $\otimes$ denotes a convolution.

The aperture function $\tilde{A}(\vec{U}_\perp, \nu)$, peaks at $\vec{U}_\perp = 0$ and has a finite width. Hence, we shall retain the second term in all subsequent discussions.

We note that, ignoring the $w - term$ leads to a simplification of the expression for visibility and in the 2D approximation, $\mathcal{V}^{2D}(\vec{U}_\perp, \nu)$ is the Fourier Transform of $A(\vec{\theta}, \nu)\delta I(\vec{\theta}, \nu)$. Hence, we have,

$$A(\vec{\theta}, \nu)\, \delta I(\vec{\theta}, \nu) = \int d^2\vec{U}'_\perp\, \mathcal{V}^{2D}(\vec{U}'_\perp, \nu)\, e^{-2\pi i . \vec{U}'_\perp \cdot \vec{\theta}} \tag{4}$$

Substituting in equation (1) we obtain

$$\mathcal{V}^{3D}(\vec{U}, \nu) = \int d^2\vec{U}'_\perp\, K(\vec{U}, \vec{U}'_\perp, \nu)\, \mathcal{V}^{2D}(\vec{U}'_\perp, \nu) \tag{5}$$

Where the kernel $K(\vec{U}, \vec{U}'_\perp, \nu)$ is given as

$$\begin{aligned} K(\vec{U}, \vec{U}'_\perp, \nu) &= \int d\Omega_{\hat{n}}\, e^{-2\pi i\ (\vec{U}'_\perp - \vec{U})\cdot(\hat{n}-\hat{k})} \\ &= 4\pi j_0(2\pi \mid \vec{U}'_\perp - \vec{U} \mid), \end{aligned} \tag{6}$$

with j_0 denoting the Spherical Bessel function.

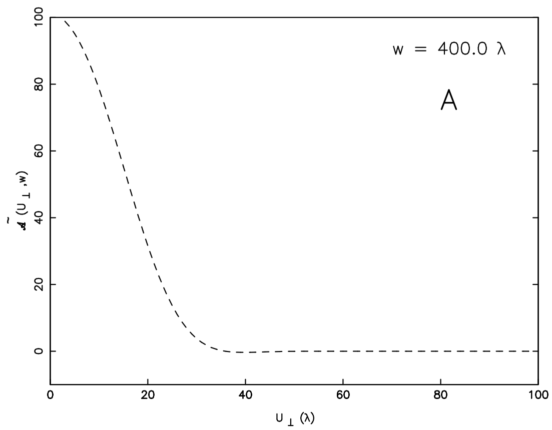

Figure 1. Modified aperture $\tilde{\mathcal{A}}(\vec{U}_{\perp}, w)$ plotted as a function of $\vec{U}_{\perp}$ for two different values of w, (A) $w = 400.\lambda$ and (B) $w = 12.$ kλ at $\nu = 150$ MHz. Solid line in B shows the gaussian envelope.

Defining a quantity $\tilde{\mathcal{A}}(\vec{U}, \nu)$ as

$$\tilde{\mathcal{A}}(\vec{U}, \nu) = 4\pi \int d^2\vec{U}'_{\perp}\; j_0(2\pi|\vec{U}'_{\perp} - \vec{U}|)\tilde{A}(\vec{U}'_{\perp}, \nu). \tag{7}$$

The 3D visibility takes the form

$$\mathcal{V}^{3D}(\vec{U}, \nu) = \int d^2\vec{U}'_{\perp}\; \tilde{\mathcal{A}}(\vec{U} - \vec{U}'_{\perp})\; \tilde{\delta I}(\vec{U}'_{\perp}) \tag{8}$$

It is to be noted that the 'w' dependence of $\mathcal{V}^{3D}(\vec{U}, \nu)$ is translated to the function $\tilde{\mathcal{A}}(\vec{U} - \vec{U}_{\perp})$. This can be regarded as a modified aperture function. We investigate the nature of the modified aperture $\tilde{\mathcal{A}}(\vec{U}_{\perp}, w)$ as a function of $U_{\perp}$ at different values of w. We have assumed the primary aperture $A(\vec{U}_{\perp})$ to be a gaussian, $\exp\left[-\frac{U_{\perp}^2}{2U_0^2}\right]$, of width U_0, and evaluated the integral in Eqn. (7) numerically. This indicates that $\tilde{\mathcal{A}}(\vec{U}, \nu)$ has an implicit dependence on U_0. For an antenna of diameter D, U_0 can be approximately written as $U_0 \sim D/\lambda$ for observing wavelength λ. We use $U_0 = D/\lambda$ with $D = 45$ m (specifications of the GMRT) for the subsequent discussion. This corresponds to a field of view of 4° at 150 MHz. We shall discuss the effect of larger field of view later.

Figure 1 shows the variation of the modified aperture $\tilde{\mathcal{A}}(\vec{U}_{\perp}, w)$ as a function of $U_{\perp}$ for two representative values of w (A: $w = 400\ \lambda$, B: $w = 12$ kλ) at frequency $\nu = 150\mathrm{MHz}$.

For $w = 0$ one reproduces the primary gaussian aperture trivially. We also note that for $w \ll U_{\perp}$ the gaussian profile is still maintained (**Figure** 1:A). However for large values of w the aperture function manifest oscillations (**Figure** 1:B). The period of these oscillations is found to be sensitive to w (decreasing as w increases). The envelope of the modified aperture is also a gaussian, $C\exp\left[-\frac{U^2}{2U_w^2}\right]$, with the parameter U_w being a measure of it's dispersion. Ignoring the effect of oscillations in $\tilde{\mathcal{A}}$, we note that the 3D formalism can be recast in the same form as it's

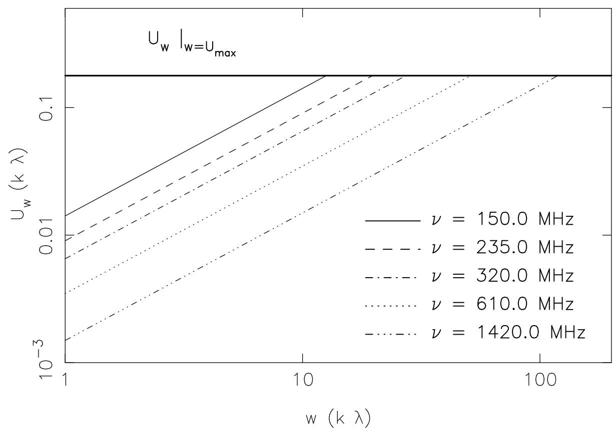

Figure 2. U_w is plotted as a function of w for different central frequencies of GMRT. Horizontal solid line corresponds to the maximum possible baseline.

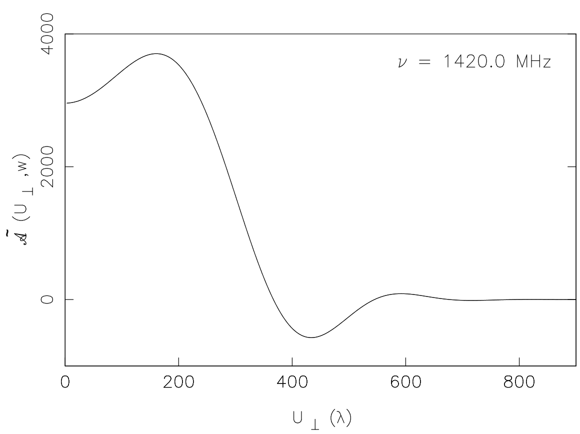

Figure 3. Modified aperture $\tilde{\mathcal{A}}(\vec{U}_\perp, w)$ plotted as a function of $\vec{U}_\perp$ for $w = 120.$ k λ at $\nu = 1420.$ MHz.

2D counterpart with U_w taking the role of U_0. Hobson & Maisinger (2002) have obtained a similar result assuming small field of view, where, they have shown that the effect of w-distortion can be considered as turning the primary beam into a complex gaussian.

We next investigate the w and frequency dependence of U_w. For a given frequency at large w, U_w is found to increase linearly with w, i.e, $U_w \sim m(\nu)w$. This implies that the $w - term$ effectively broadens the aperture of the instrument. **Figure** 2 shows the variation of U_w with w for different frequencies in log-log scale. We note that, the slope $m(\nu)$ (as represented by the y-intercept in the **Figure** 2) determines the effect of $w - term$ for increasing values of w. A small value of $m(\nu)$ implies a slow increase of U_w with w and the effect of the $w - term$ is less. $m(\nu)$ is found to fall off as $\sim 1/\nu$ with frequency. This indicates that the departure from the flat-sky approximation is more pronounced at the lower frequencies. Redshifted 21 cm line observed at

frequency ν probes the redshift $z = \left[\frac{1420(\mathrm{MHz})}{\nu} - 1\right]$. Hence, one may expect the $w-term$ to have a greater effect while probing higher red-shifts.

21 cm line has been used to study the ISM dynamics of the nearby galaxies $(z \sim 0)$. **Figure** 3 shows the modified aperture function for frequency 1420 MHz and $w = 120$ kλ (this being the maximum U for GMRT like arrays). At this frequency, for GMRT, $U_0 = 0.1\mathrm{k}\lambda$, whereas $U_w \mid_{w=U_{max}} = 0.18\mathrm{k}\lambda$. It follows that the flat-sky approximation can be safely used if the largest length scale probed, corresponds to a $U_{\perp} \gg 3\, U_w \mid_{w=U_{max}}$.

Till now, we have investigated the effect of $w - term$ using $D = 45$ m, which corresponds to an field of view of 4° at 150 MHz. We estimated $U_w \mid_{w=U_{max}}$ assuming $D = 4$ m to $D = 45$ m at $\nu = 150$ MHz. At $D = 4$ m, (which corresponds to the largest proposed field of view, 45°, of SKA), $U_w \mid_{w=U_{max}} = 4$ kλ for $U_{max} = 120$ kλ. We have also observed that, $U_w \mid_{w=U_{max}} \sim 1/D$. This indicates that the effect of $w-term$ is more pronounced for larger field of view, as expected.

VISIBILITY CORRELATION AND POWER SPECTRUM ESTIMATION

The power spectrum, $P(U_{\perp}, \Delta\nu)$ of the intensity fluctuations δI on the sky is defined as

$$\langle \tilde{\delta I}(\vec{U}_{\perp}, \nu_1)\, \tilde{\delta I}(\vec{U}'_{\perp}, \nu_2)\rangle = P(U_{\perp}, \Delta\nu)\, \delta^2_{\mathbf{D}}(\vec{U}_{\perp} - \vec{U}'_{\perp}) \tag{9}$$

where $\Delta\nu = |\nu_1 - \nu_2|$. We shall be considering, for simplicity, $\nu_1 = \nu_2$ in all subsequent discussions and hence drop the $\Delta\nu$ dependence of the power spectrum.

We define

$$V_2^{3D}(\vec{U}_a, \vec{U}_b) = \langle \mathcal{V}^{3D}(\vec{U}_a)\, \mathcal{V}^{3D*}(\vec{U}_b)\rangle. \tag{10}$$

Using Eqn. (8) we obtain

$$\begin{aligned} V_2^{3D}(\vec{U}_a, \vec{U}_b) = &\int d^2\vec{U}''_{\perp} \int d^2\vec{U}'_{\perp} \tilde{\mathcal{A}}(\vec{U}_a - \vec{U}''_{\perp}) \\ &\tilde{\mathcal{A}}^*(\vec{U}_b - \vec{U}'_{\perp}) \langle \tilde{\delta I}(\vec{U}''_{\perp})\, \tilde{\delta I}^*(\vec{U}'_{\perp})\rangle \end{aligned} \tag{11}$$

Using the definition of the power spectrum and considering the correlation at the same base-line $\vec{U}$, this simplifies to

$$V_2^{3D}(U) = \int d^2\vec{U}'_{\perp}\, |\tilde{\mathcal{A}}(\vec{U} - \vec{U}'_{\perp})|^2\, P(U'_{\perp}) \tag{12}$$

Noting that the effect of the $w - term$ is contained in the modified aperture $\tilde{\mathcal{A}}$ we can retrieve the 2D estimator V_2^{2D} used earlier Dutta et al. (2009a) by replacing $\tilde{\mathcal{A}}$ with $\tilde{A}$. Hence,

$$V_2^{2D}(U_{\perp}) = \int d^2\vec{U}'_{\perp}\, |\tilde{A}(\vec{U}_{\perp} - \vec{U}'_{\perp})|^2\, P(U'_{\perp}). \tag{13}$$

We shall now discuss the effect of the $w - term$ on the estimator defined in Eqn. (12).

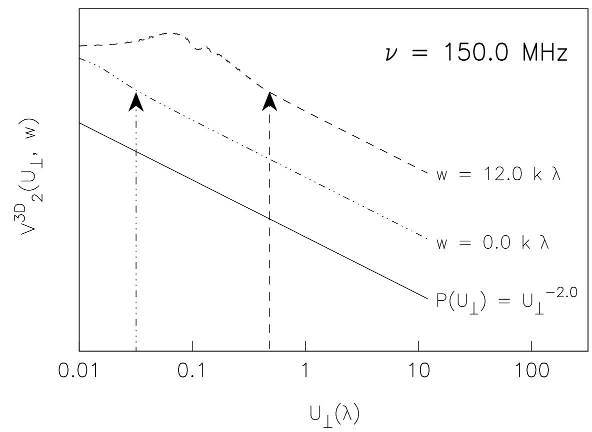

Figure 4. V_2^{3D} as a function of $U_\perp$ for $w = 0$ (dot-dash) and $w = U_{max}$ (dash) at $\nu = 150$ MHz, assuming $P(U_\perp) = {U_\perp}^{-2}$. We also plot $P(U_\perp) = {U_\perp}^{-2}$ (solid line) for reference. The vertical arrows show the $U_\perp$ value above which the power law is recovered at 1%. Note that the plots are given arbitrary offset for clarity.

Figure 4 shows V_2^{3D} plotted as a function of $U_\perp$ for two values of w, ($w = 0$ and $w = U_{max}$) at $\nu = 150$ MHz, assuming $P(U_\perp) = {U_\perp}^{-2}$. We have chosen $U_{max} = 12k\lambda$ (this being the largest baseline for the GMRT at 150 MHz). We have shown the power spectrum $P(U_\perp) = {U_\perp}^{-2}$ for comparison. For large values of w, V_2^{3D} show oscillations for $U_\perp < U_w$, which arises due to the oscillatory nature of $|\tilde{\mathcal{A}}(U)|^2$.

We find that V_2^{3D} faithfully recovers the power law ${U_\perp}^{-2}$ (at 1%) for $U_\perp$ greater than a certain value. This value is found to be $3 \times (\sqrt{2}U_0)$ for $w = 0$ and $3 \times (\sqrt{2}U_w)$ at $w = 12$ kλ. Hence, a non-zero $w - term$ changes the $U_\perp$ value beyond which the power spectrum estimation would be valid.

The quantity of interest in power spectrum estimation using the the radio interferometric observations used earlier (Bharadwaj & Ali, 2005; Dutta et al., 2009a) is

$$\mathcal{E}(U_\perp) = \int_0^{U_{max}} dw \; \rho(w) \; V_2^{3D}(U_\perp, w), \tag{14}$$

where, $\rho(w)$ is a normalized probability distribution of w. The function $\rho(w)$ is specific to an observation as well as to the array configuration of the interferometer. Hence, it is difficult to make a general quantitative statement regarding the effect of $w - term$ in $\mathcal{E}$. Since, for a given w, the largest baseline above which $V_2^{3D}(U_\perp, w) \sim P(U_\perp)$ is U_w, we can qualitatively state that $\mathcal{E}$ gives a good estimation of the power spectrum for $U \geq 3 \; U_w \; |_{w=U_{max}}$. It is important to note that, for a specific array configuration, $U_w \; |_{w=U_{max}}$ is independent of the frequency ν, whereas $U_{max} \propto \nu$ (**Table** 1). Hence, the $U_\perp$ range amenable for power spectrum estimation is larger for large observing frequencies.

	$w = 0$	$w \sim U_{max}$
Aperture	$\tilde{A}(U_\perp)$ width U_0	$\tilde{\mathcal{A}}(U_\perp, w)$ width U_w
Visibility correlation	$V_2^{2D}(U_\perp)$	$V_2^{3D}(U_\perp, w)$

Table 1. The effect of $w - term$, comparison between various quantities.

	150 (M Hz)	1420 (M Hz)
U_0	0.01	0.1
U_{max}	12.0	120.0
$U_w \mid_{w=U_{max}}$	0.18	0.18

Table 2. Relevant $U_\perp$ (k λ) values at different frequencies.

3 DISCUSSION AND CONCLUSION

We have studied the effectiveness of the widely used power spectrum estimator in presence of a non zero w. The $w - term$ is found to affect the visibility correlation estimator through a modification of the aperture function (**Table** 1). The effect is more pronounced for the lower frequencies, where, for a given interferometer, the baseline range over which V_2^{3D} reproduces the power spectrum, is reduced. This restricts the largest possible length scales that can be probed using the 21 cm radiation from the epoch of reionization ($20 \lesssim z \lesssim 6$). However, for the observations of the nearby universe (i.e, $\nu = 1420$ MHz), $w - term$ effect is not too significant (**Table** 2).

The two-visibility correlation defined in Eqn. (12) has a positive noise bias associated with it, which can even exceed the signal in radio interferometric observations. To reduce the effect of the noise bias, one may follow the method used in the 2D analysis (Begum et al., 2006; Dutta et al., 2008, 2009a) considering correlation of the visibilities at two nearby baselines (Eqn. (11), with $\vec{U}_b = \vec{U}_a + \Delta\vec{U}$). For simplicity, we have considered visibility correlations at the same baseline (Eqn. (12)). However, we note that an alternative method for reducing the noise bias is to observe the same field repeatedly for the same baseline configuration.

Redshifted 21 cm observations allow us to probe the HI distribution of the universe over continuously varying redshifts by tuning the frequency of the radio observations. By correlating visibilities at different frequencies it is possible to do a tomographic study of the HI distribution, thereby probing the 3D power spectrum. In this work we have restricted ourselves to visibility correlations at the same frequency of observation. Future investigations may consider the effect of a nonzero $\delta\nu$ on power spectrum estimation.

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